On the Parameterized Complexity of Graph Bisections

René van Bevern[1] Andreas Emil Feldmann[2] Manuel Sorge[1]
Ondřej Suchý[3]

[1]Institut für Softwaretechnik und Theoretische Informatik, TU Berlin, Germany

[2]Combinatorics & Optimization, University of Waterloo, Canada

[3]Faculty of Information Technology, Czech Technical University in Prague

WG 2013

Motivation

Task I (Parallel Computation) Equally separate tasks between two processors, minimizing inter-processor communication

Task II (Divide and Conquer) Separate the instance of a graph problem into equally large subproblems with few edges between them

Formally: the Bisection Problem

Bisection

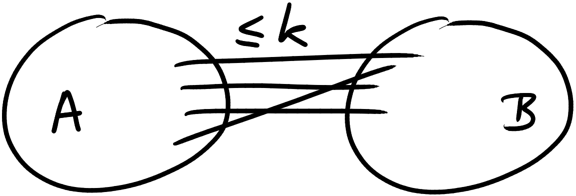
Input: A graph $G = (V, E)$ and a positive integer k.

Question: Does G have a bisection with cut size at most k?

Herein,

bisection: a partition $A \uplus B = V$ such that $|A|, |B| \leq \lceil |V|/2 \rceil$

cut size: number of edges between vertices in A and in B



Known Results

NP-hard *[Garey, Johnson, and Stockmeyer, TCS'76]*

Optimal cut size $O(\log n)$-approximable, but *[Räcke, STOC'08]*
not constant-factor (under Unique Games Conjecture)
[Khot and Vishnoi, FOCS'05]

Polynomial-time solvable on trees *[MacGregor, PhD thesis, 1978]*
and on solid grid graphs *[Feldmann and Widmayer, ESA'11]*

Solvable in $O(22.7^k n^3)$ time on planar graphs, NP-hardness open
[Bui and Peck, SICOMP'92]

Solvable in $f(\omega)\,\mathrm{poly}(n)$ time on graphs of treewidth ω
[Soumyanath and Deogun, CGTC'90; Wiegers, MFCS'90]

Parameterized Complexity of Bisection

NP-hard, hard to approximate $\rightsquigarrow$ parameterized complexity

A **parameterized problem** is a language $L \subseteq \Sigma^* \times \mathbb{N}$.

A parameterized problem L is **fixed-parameter tractable (FPT)** with respect to some **parameter k** if $(x,k) \in L$ is decidable in $f(k)\,\mathrm{poly}(n)$ time.

A parameterized problem L has a **polynomial kernel** if there is a polynomial-time computable function $f\colon \Sigma^* \times \mathbb{N} \to \Sigma^* \times \mathbb{N}$ such that $(x,k) \in L \iff f(x,k) \in L$ and $|f(x,k)| \leq \mathrm{poly}(k)$.

Our Results

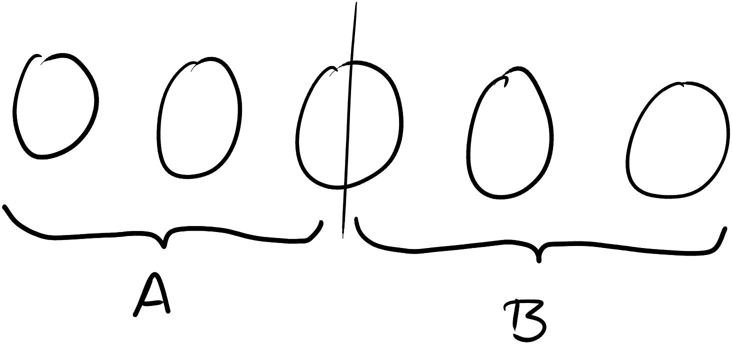
Bisection

- has no polynomial kernel w. r. t. parameter “bandwidth”, “cut size”, “pathwidth”, “cliquewidth”, unless coNP $\subseteq$ NP/poly
- is FPT w. r. t. parameter “distance to cliquewidth-q” if q is constant and the cliquewidth-q vertex deletion set is given or computable in FPT time

More general problem **Vertex Bisection** is

- W[1]-hard, i. e. unlikely to be FPT w. r. t. combined parameter “cut size” and “number of resulting components”
- FPT w. r. t. “cut size” for constant number of resulting components

No Polynomial Kernel



Distance to Constant Cliquewidth

Theorem

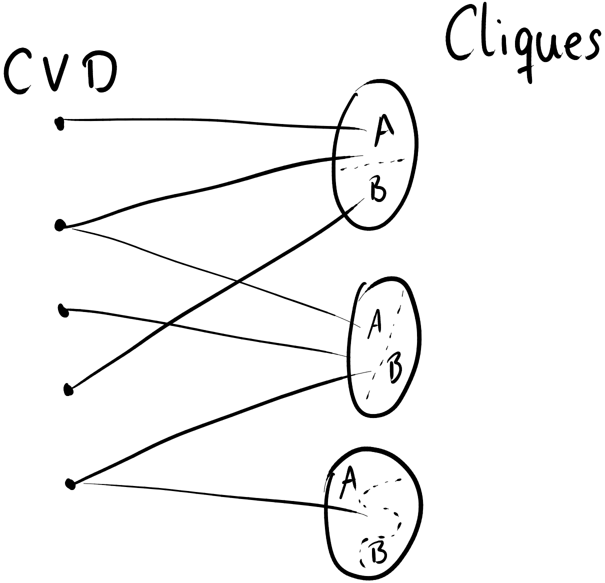
Bisection is FPT w. r. t. parameter "distance to cliquewidth-q" if q is constant and the cliquewidth-q vertex deletion set is given or computable in FPT time.

"Distance to cliquewidth-q" generalizes many previously considered parameters.

FPT-time computable cliquewidth-q deletion sets include

- cluster vertex deletion set ($q = 2$),
- feedback vertex set ($q = 3$), *[Guo et al., JCSS'07; Dehne et al., TCS'07]*
- distance to treewidth-t graphs, *[Fomin et al., FOCS'12]*
- cograph deletion set ($q = 2$).

Example: Cluster Vertex Deletion Set



Cut Size and Number of Cut out Components

In practice, the optimum cut usually cuts the input graph into very few (mostly two) connected components.

[Bui, Chaudhuri, Leighton, and Sipser, Combinatorica 1987]

[Peter Arbenz, ETH Zürich, personal communication]

[Delling, Goldberg, Razenshteyn, and Werneck, ALENEX'12]

[Renato Werneck, Microsoft Research, personal communication]

Vertex Bisection

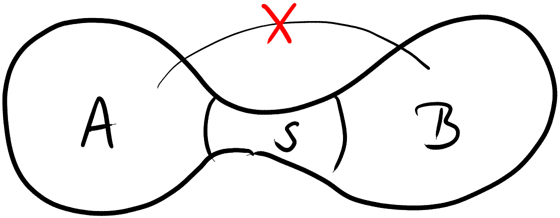
We show results w. r. t. parameter "cut size" for a constant number of cut out components for a more general problem:

Vertex Bisection

Input: A graph $G=(V,E)$ and a positive integer k.

Question: Does G have a balanced separator of size at most k?

A **balanced seperator** S partitions V into vertex sets S,A,B such that there are no edges between A and B in G, and $||A|-|B|| \leq 1$.



Parameterized Complexity of Vertex Bisection

Theorem. **Vertex Bisection** is W[1]-hard w. r. t. separator size and number of resulting components.

Theorem. Given a **Vertex Bisection** instance (G, k) and a non-negative integer c, in $h(c, k) \cdot n^{c+3}$ time we can

- either find a balanced separator for G of size at most k or
- reveal that no size-k balanced separator S exists such that removing S leaves at most c connected components.

Lemma. There is a reduction from **Bisection** to **Vertex Bisection** that increases

- the desired cut/separator size and
- the number of resulting connected components

only by a constant.

Question. Is (Edge) **Bisection** FPT w. r. t. cut size?

FPT Algorithm for Vertex Bisection

Central observations:

- balanced separators consist of minimal s-t-separators, for s and t being vertices from some set T of “terminal vertices”: one terminal vertex for each connected component in the bisected graph
- **Vertex Bisection** is FPT w. r. t. treewidth

Algorithm:

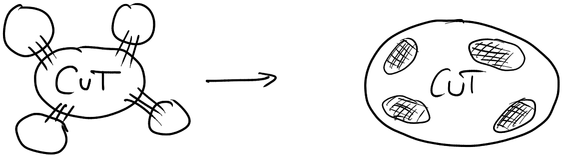
1. Guess terminal vertex set T of c vertices ($O(n^c)$)
2. Reduce treewidth to $h(c, k)$ under preservation of all minimal s-t-cuts for $s, t \in T$
3. Use treewidth algorithm for **Vertex Bisection**

Treewidth Reduction

Theorem (Marx et al., STACS'10)

Let $G = (V, E)$ be a graph, $T \subseteq V$, and $k \in \mathbb{N}$, let C be the set of all vertices participating in a minimal s-t-separator for $s, t \in T$. In $f(k, |T|)\,\mathrm{poly}(n)$ time one can compute a graph G^* such that:

- $C \cup T \subseteq V(G^*)$.
- *For $s, t \in T$, $K \subseteq V(G^*)$ with $|K| \leq k$ is a minimal s-t-separator in G^* iff $K \subseteq C \cup T$ and it is a minimal s-t-separator in G.*
- *the treewidth of G^* is at most $h(k, |T|)$.*



Question: is it possible to generalize this theorem to handle global restrictions such as balance requirements?

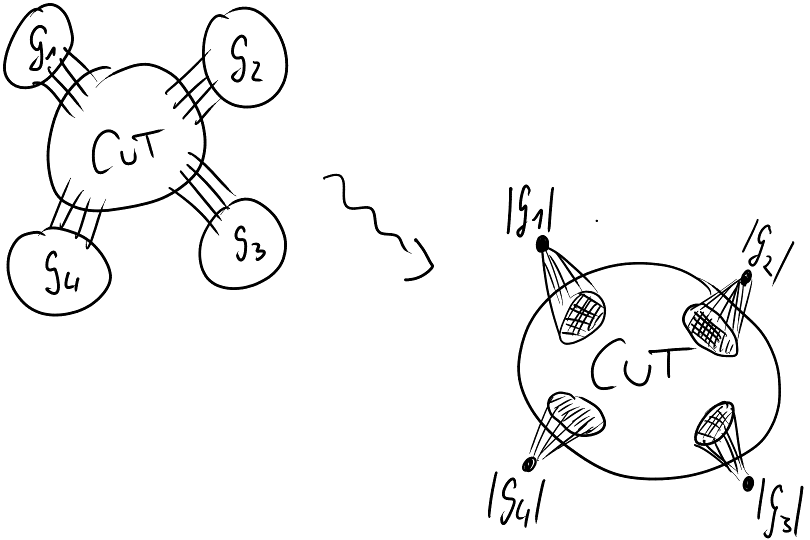
[Marx, O'Sullivan, Razgon, arXiv 2011]

Treewidth Reduction for Global Constraints

Let $G = (V, E)$ be a graph, $T \subseteq V$, and $k \in \mathbb{N}$, let C be the set of all vertices participating in a minimal s-t-separator for $s, t \in T$. In $f(k, |T|)n^2$ time one can compute a graph G^* and vertex weights λ such that:

- $C \cup T \subseteq V(G^*)$.
- Any A'-B'-separator S' in G^* corresponds to a A-B-separator S in G with $\lambda(A') = |A|$, $\lambda(B') = |B|$, $\lambda(S') = |S|$. If S' is a minimal s-t-separator for $s, t \in T$, then so is S.
- For $s, t \in T$, a minimal s-t-separator S in G with $|S| \leq k$ is also a minimal s-t-separator S' in G^*. If S is an A-B-separator, then S' is an A'-B'-separator S' with $\lambda(A') = |A|$, and $\lambda(B') = |B|$.
- The treewidth of G^* is at most $g(k, |T|)$.

Example: Treewidth Reduction



Conclusion

Conceptual Contributions.

Cliquewidth-q deletion set as generalization of many graph parameters

Generalization of Marx et al.'s treewidth reduction technique for weighted graphs and global constraints

Main Open Questions.

Decide whether **Bisection** is FPT w. r. t. cut size
Observe that **Vertex Bisection** is W[1]-hard

Turn our classification results into practical algorithms