# Linear-Time Computation of a Linear Kernel for Dominating Set on Planar Graphs

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joint work with

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# Dominating Set

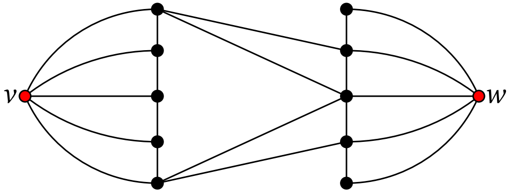
**Dominating Set**
**Input:** A graph $G = (V, E)$ and a natural number $k$.
**Question:** Is there a dominating set $D \subseteq V$ with $|D| \leq k$ for $G$?

Here, $D$ is a dominating set $\iff V \subseteq N[D]$.

Size of a minimum dominating set for $G$ is denoted by $\gamma(G)$.

Dominating Set is NP-hard even when restricted to planar graphs.

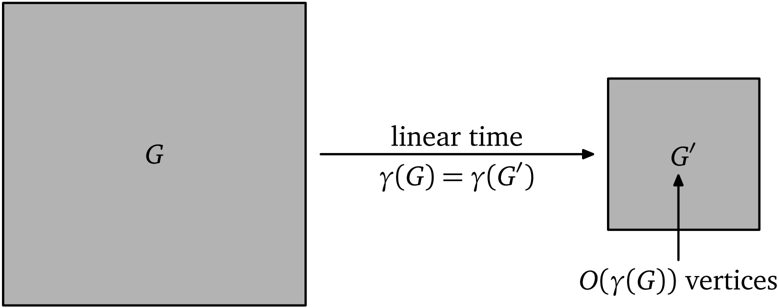
# Example for Dominating Set




The set $\{v, w\}$ dominates the shown graph.

# Problem Kernels

We show an $O(\gamma(G))$-vertex problem kernel for DOMINATING SET on planar graphs that is computable in linear time.



# Why Problem Kernels Matter

In general, problem kernels yield equivalent smaller instances in polynomial time. Hence, used as a preprocessing step, they can

- speed up algorithms that exactly solve NP-hard problems,
- speed up slow but effective approximation algorithms,
- speed up slow but effective kernelization algorithms,
- speed up heuristics often used in practice.

Combining fast kernelization algorithms with effective ones yields small kernels fast.

In contrast, it is unclear how a fast approximation algorithms can be combined with an effective one to get an approximation algorithm that is both fast and effective.

# The Kernel Size Race

Problem kernels are currently the hottest topic in parameterized algorithmics, resulting in a kernel size race.

## An Example:

**Feedback Vertex Set**: delete at most $k$ vertices to transform a graph into a forest (make it cycle-free).

Burrage, Estivill-Castro, Fellows, and Langston, IWPEC 2006:
$O(k^{11})$-vertex kernel

Bodlaender and van Dijk, TCS 2010:
$O(k^3)$-vertex kernel

Thomassé, ACM Trans. Algorithms 2010:
$O(k^2)$-vertex kernel

# The Kernel Size Race

Problem kernels are currently the hottest topic in parameterized algorithmics, resulting in a kernel size race.

## Another Example:

**CLUSTER EDITING**: delete or add at most $k$ edges to transform a graph into a disjoint union of cliques.

Gramm, Guo, Hüffner, and Niedermeier, TCS 2005:
$O(k^2)$-vertex kernel

Fellows, Langston, Rosamond, and Shaw, FCT 2007:
$O(k)$-vertex kernel

Guo, TCS 2009:
$4k$-verex kernel

Chen and Meng, COCOON 2010:
$2k$-vertex kernel

# Problem Kernels for Dominating Set

## Focusing on Kernel Size:

Alber, Fellows, and Niedermeier, J. ACM, 2004:
$335\gamma$-vertex kernel in $\mathrm{O}(n^3)$ time on planar graphs.

Chen, Fernau, Kanj, and Xia, SIAM J. Comput., 2007:
$67\gamma$-vertex kernel in $\mathrm{O}(n^3)$ time on planar graphs.

## Focusing on Larger Graph Classes:

Fomin and Thilikos, ICALP 2004:
$O(\gamma + g)$-vertex kernel in $\mathrm{O}(gn^3)$ time on graphs of genus $g$.

Philip, Raman, and Sikdar, ESA 2009:
$O(\gamma^{2(d+1)^2})$-vertex kernel in $\mathrm{O}(2^d dn^2)$ time on $d$-degenerate graphs.

Fomin, Lokshtanov, Saurabh, and Thilikos, SODA 2010:
$O(\gamma)$-vertex kernel in polynomial time on apex-minor free graphs.

# Linear-Time Problem Kernel for Dominating Set

## Focusing on Running Time:

van Bevern, Hartung, Kammer, Niedermeier, Weller, Manuscript 2011 (submitted):
O($\gamma$)-vertex kernel in O($n$) time.

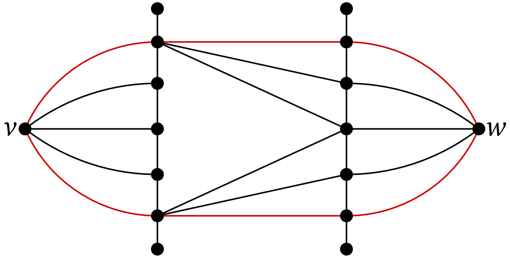
## Small Kernel Fast:

67$\gamma$-vertex kernel in O($\gamma^3 + n$) time using the above kernelization as preprocessing step for the following result:

Chen, Fernau, Kanj, Xia, SIAM J. Comput., 2007:
67$\gamma$-vertex kernel in O($n^3$) time on planar graphs.

# Tools for Obtaining Dominating Set Problem Kernels

Many results exploit a result by Alber, Fellows, and Niedermeier, J. ACM 2004: a graph is decomposable into $O(\gamma)$ *regions*:

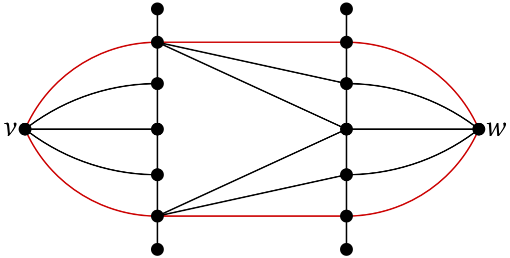



A *region* $R(v,w)$ between two vertices $v$ and $w$ is a closed bounded subset of the plane such that:

1. the boundary of $R(v,w)$ is formed by two simple paths between $v$ and $w$, each of which has length at most three and
2. all vertices inside the region $R(v,w)$ are from $N[v] \cup N[w]$.

# Problem Kernel Analysis of Alber et al., J. ACM, 2004

A graph is decomposable into $O(\gamma)$ regions:




Problem kernel is obtained as follows:

1. shrink each region to $O(1)$ vertices and
2. reduce number of vertices outside of regions to $O(\gamma(G))$.

# Our Problem Kernel

Re-use the region framework and data reduction rules by Alber et al. with slight modifications, but find and update the structures to be reduced in linear time.

## Example: Finding and Shrinking Regions

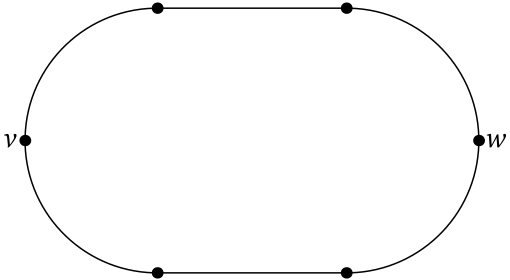
Problem: graph is decomposable into $O(\gamma)$ regions, but we do not actually *have* them $\rightsquigarrow$ shrink everything that *could* be a region.

Alber et al. delete vertices from $N[v] \cup N[w]$ for all vertices $v, w$.
$\rightsquigarrow$ $\Omega(n^2)$ time. Rules applied $\mathrm{O}(n)$ times $\rightsquigarrow$ $\mathrm{O}(n^3)$ time.

Assuming that we know how to shrink regions: how to find them?

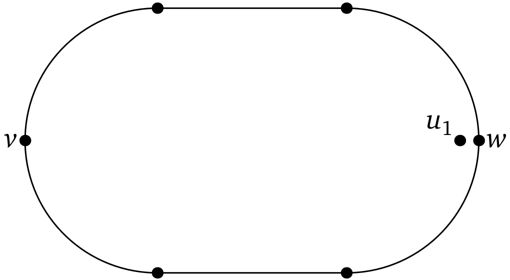
- Largely exploit the restricted structure of regions and
- know when to stop: do not apply data reduction rules more often than necessary to obtain the problem kernel.

# Shrinking Regions in Linear Time



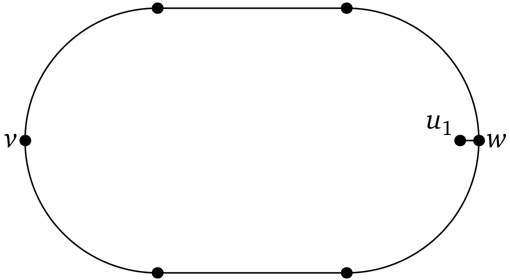

Consider this region $R(v, w)$.

# Shrinking Regions in Linear Time



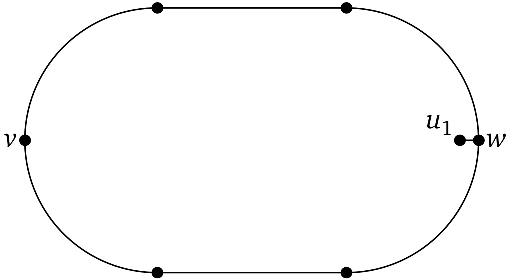

Assume that $u_1 \in R(v, w)$. Then, $u_1 \in N[v, w]$.

# Shrinking Regions in Linear Time



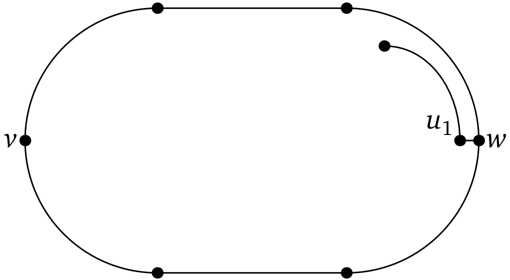

W. l. o. g., assume that $u_1 \in N(w)$.

# Shrinking Regions in Linear Time



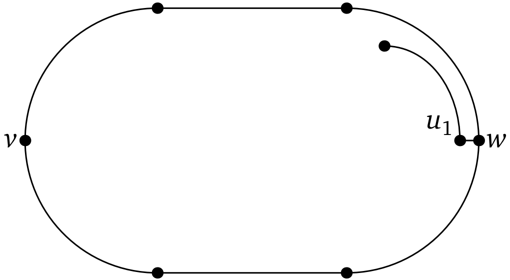

If $\deg(u_1) \leq 1$: delete $u_1$ and remember $w$ to be dominating.

# Shrinking Regions in Linear Time



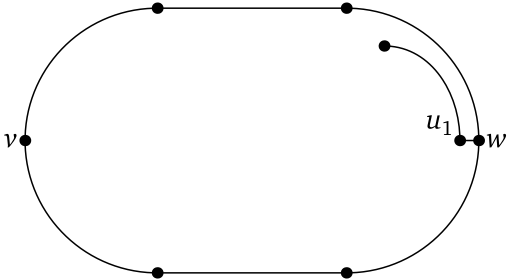

Hence, $u_1$ has a neighbor, which, due to planarity, is in $R(v, w)$.

# Shrinking Regions in Linear Time



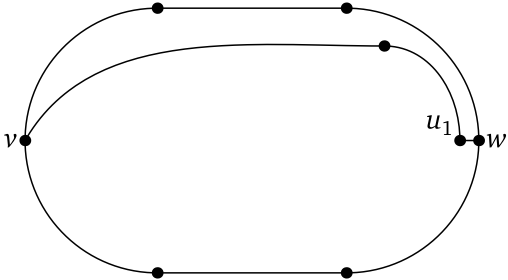

If $N[N[u_1]] \subseteq N[w]$, delete $u_1$ (remember $w$ to be dominating).

# Shrinking Regions in Linear Time



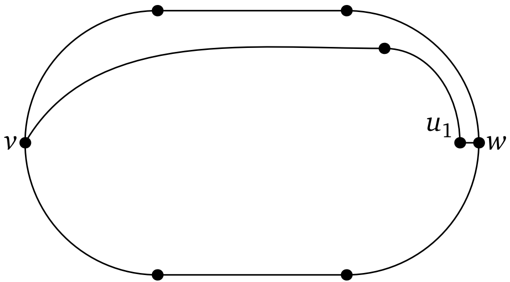

Assume that $u_1$'s neighbor is nonadjacent to $w$.

# Shrinking Regions in Linear Time



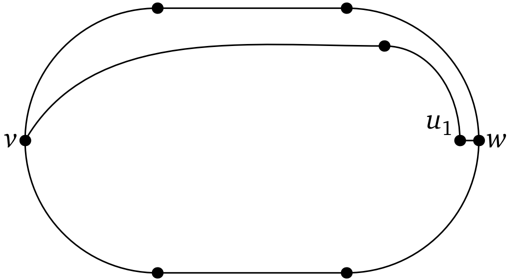

Then, $u_1$'s neighbor is adjacent to $v$.

# Shrinking Regions in Linear Time



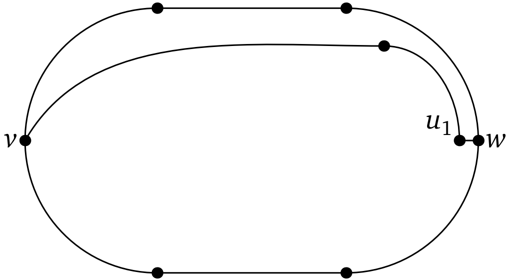

Observation: inner vertices of $R(v, w)$ lie on short $v$-$w$-paths.

# Shrinking Regions in Linear Time



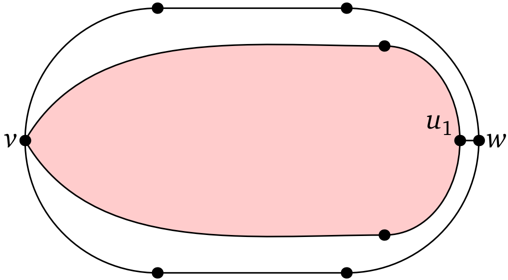

Can quickly list such paths if vertices of $R(v, w)$ have small degree.

# Shrinking Regions in Linear Time



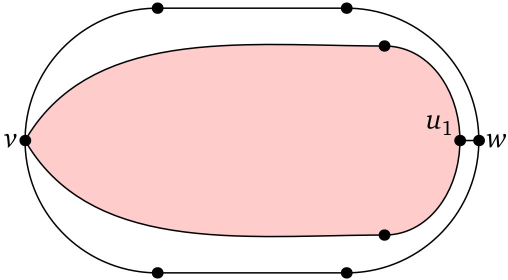

If $u_1$ has large degree, then $N(u_1) \cap N(v)$ or $N(u_1) \cap N(w)$ is large.

# Shrinking Regions in Linear Time



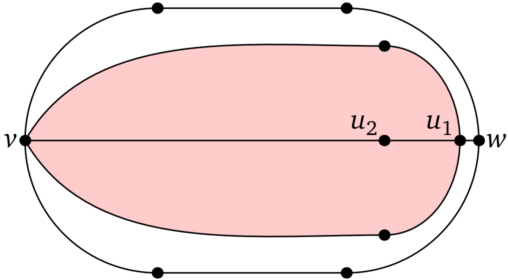

Assume that $N(u_1) \cap N(v)$ is large $\rightsquigarrow$ red sub-region $R(v, u_1)$.

# Shrinking Regions in Linear Time



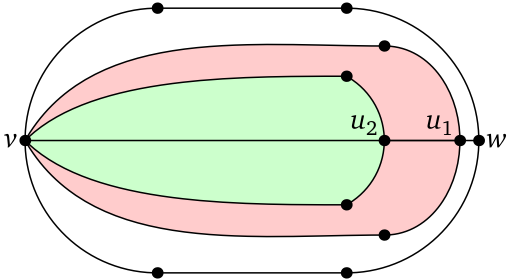

Can shrink $R(v, u_1)$ if it only contains vertices with small degree.

# Shrinking Regions in Linear Time



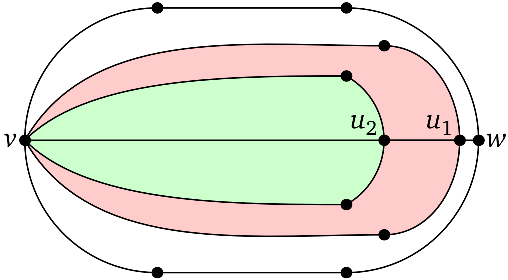

Assume that $u_2 \in R(v, u_1)$. By planarity, $|N(u_2) \cap N(1)| \in O(1)$.

# Shrinking Regions in Linear Time



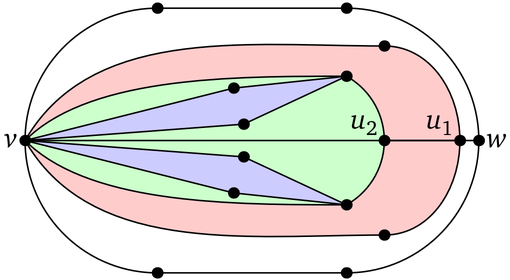

Hence, if $u_2$ has large degree, then $N(u_2) \cap N(v)$ (green) is large.

# Shrinking Regions in Linear Time



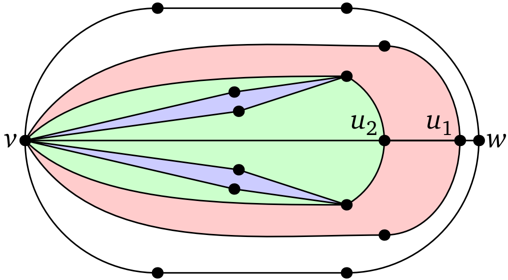

Can shrink $R(v, u_2)$ if it only contains vertices with small degree.

# Shrinking Regions in Linear Time



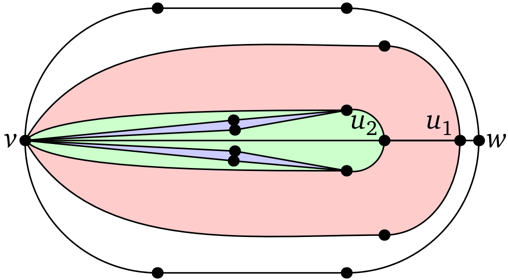

This is the case since vertices in blue regions are safely deletable.

# Shrinking Regions in Linear Time



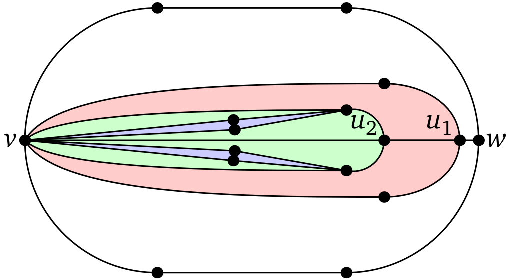

Shrink blue regions $\rightsquigarrow$ vertices in $R(v, u_2)$ have low degree.

# Shrinking Regions in Linear Time



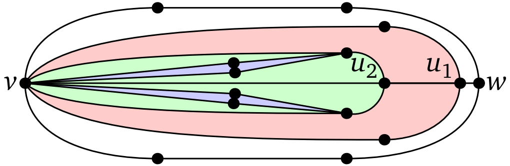

Shrink $R(v, u_2) \rightsquigarrow$ vertices in $R(v, u_1)$ have low degree.

# Shrinking Regions in Linear Time




Shrink $R(v, u_1) \rightsquigarrow$ vertices in $R(v, w)$ have low degree.

# Shrinking Regions in Linear Time




We can now find and shrink $R(v, w)$.

# Shrinking Regions in Linear Time

Three times, basically apply the following algorithm:

1. Preprocess (e. g. degree-one deletion rule)
2. Find deletion candidates regions:
   - **2.1** For each $v \in V$, in $\mathrm{O}(\deg(v))$ time, list all short paths starting at $v$, having length at most four, and only crossing vertices with constant degree.
   - **2.2** For each such path $p$, let $v_p$, $w_p$ be its endpoints.
   - **2.3** If $p$ contains only vertices of $N[v_p] \cup N[w_p]$, then $p$ is a deletion candidate (could be part of a region to shrink).
3. Shrink regions, which contain $\mathrm{O}(n)$ deletion candidates in total, since only $\sum_{v \in V} \deg(v) = \mathrm{O}(n)$ paths are generated.

# Conclusion

Not only the size of problem kernels, but also the running time of kernelization algorithms has to be optimized.

Optimize one of size and speed; get the other for free:

- can combine speed-optimized and size-optimized kernelization algorithms to obtain small kernels fast

⇝ $67\gamma$-vertex kernel in $O(\gamma^3 + n)$ time for Dominating Set in planar graphs.[1]

⇝ faster algorithms for solving hard problems.

[1]Executing the algorithm by Chen et al. (SIAM J. Comput., 2007) after ours